SYSTEMS OF LINEAR INEQUALITIES.

By Walter B. Carver.

In a paper under this same title,* Professor L. L. Dines found a necessary and sufficient condition for the existence of solutions of a system of linear inequalities, for both the homogeneous and non-homogeneous cases. His condition was expressed in terms of the "I-rank" of the matrix. It is the purpose of the present paper to give, in a quite different form, a necessary and sufficient condition for the *non-existence* of solutions; and to consider the questions of the independence of a system and the equivalence of two systems.

Let S represent the system of m linear inequalities in n variables,

$$\sum_{j=1}^{n} \alpha_{ij} x_j + \beta_i > 0, \qquad i = 1, 2, \cdots m,$$

in which the β's may or may not all be zero. For brevity we may write

$$L_i(x) \quad \text{for} \quad \sum_{j=1}^{n} \alpha_{ij} x_j + \beta_i \quad \text{and} \quad L_i'(x) \quad \text{for} \quad \sum_{j=1}^{n} \alpha_{ij} x_j.$$

The matrix of the coefficients, $\| \alpha_{ij} \|$ (not including the β's), will be denoted by M.

A system of inequalities will be said to be *consistent* or *inconsistent* according as solutions of the system do or do not exist. A single inequality will be inconsistent only when

$$\alpha_{i1} = \alpha_{i2} = \cdots = \alpha_{in} = 0, \qquad \text{and} \qquad \beta_i \lesseqqgtr 0.$$

Theorem 1. *If for a system S the rank of the matrix M is m, the system is consistent.*

We may suppose that the non-vanishing determinant of order m in the matrix M is made up of the first m columns of the matrix; and consider the set of equations,

$$\sum_{j=1}^{m} \alpha_{ij} x_j = c_i, \qquad i = 1, 2, \cdots m.$$

Since the determinant of the coefficients does not vanish, solutions of this set of equations exist for any values of the c's. Fix c's satisfying the relations $c_i > -\beta_i$, and let $a_1, a_2, \cdots a_m$ be the solution of the resulting set of equations. Then evidently $a_1, a_2, \cdots a_m, 0, \cdots 0$ is a solution of the system S of inequalities.

* These Annals, vol. 20, p. 191.

A system S of m inequalities will be said to be *irreducibly inconsistent* when the system S is inconsistent, but each sub-system of $m-1$ inequalities in S is consistent; i.e., when the omission of any one inequality from the inconsistent system leaves a consistent system. A single inequality will be irreducibly inconsistent if it is inconsistent.

THEOREM 2. *If the system S is irreducibly inconsistent, there exists a set of constants $k_1, k_2, \cdots k_{m+1}$, homogeneously unique, such that*

$$\sum_{i=1}^{m} k_i L_i(x) + k_{m+1} \equiv 0,$$

*$k_1, k_2, \cdots k_m$ being positive and k_{m+1} positive or zero; and the rank of the matrix M must be $m-1$.**

By hypothesis there exists a set of numbers $a_1, a_2, \cdots a_n$ or, briefly, a point† a, which satisfies all the inequalities except the first one. This may be conveniently expressed by saying that there exists a point a which gives the row of m symbols

$$\bar{0} \quad + \quad + \quad + \quad \cdot \quad \cdot \quad \cdot \quad +;$$

the double symbol "$\bar{0}$" indicating that $L_1(a)$ is either negative or zero, and the following plus signs indicating that each of the expressions $L_i(a)$, for $i \neq 1$, is positive. Similarly, there exists a point giving each of the rows

$$\begin{array}{cccccccc} + & \bar{0} & + & + & \cdot & \cdot & \cdot & +, \\ + & + & \bar{0} & + & \cdot & \cdot & \cdot & +, \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot, \\ + & + & + & + & \cdot & \cdot & \cdot & \bar{0}. \end{array}$$

If h_1 and h_2 are any two positive numbers whose sum is unity, we may speak of the point $h_1 a + h_2 b$ (i.e., the set of numbers $h_1 a_1 + h_2 b_1$, $h_1 a_2 + h_2 b_2$, $\cdots h_1 a_n + h_2 b_n$) as a point *between* a and b. Since the expressions $L_i(x)$ are linear, $L_i(h_1 a + h_2 b) = h_1 L_i(a) + h_2 L_i(b)$. Suppose now that a point b should exist which, when substituted in the L's, makes at least one of them positive and all of them either positive or zero; giving, for instance,

$$+ \quad 0 \quad 0 \quad + \quad \cdot \quad \cdot \quad \cdot \quad +.$$

Since there is a point a which gives

$$\bar{0} \quad + \quad + \quad + \quad \cdot \quad \cdot \quad \cdot \quad +,$$

* The method of proof of this theorem was suggested to the author by Professor Hurwitz.

† Whether the system S is or is not homogeneous, the set of numbers indicated by the phrase "the point a" will not be a homogeneous set; i.e., the point a does not mean the set of numbers $ca_1, ca_3, \cdots ca_n$.

it is evident that there would be a point between a and b which would make the L's all positive. But this is contrary to the hypothesis that the system S is inconsistent. Hence every point which, when substituted in the L's, makes at least one of them positive, will also make at least one of them negative. It follows that where we used the double symbol "$\bar{0}$" above, the zero can not occur; and that there are therefore points giving each of the rows

$$\begin{array}{cccccccc} - & + & + & + & \cdot & \cdot & \cdot & +, \\ + & - & + & + & \cdot & \cdot & \cdot & +, \\ + & + & - & + & \cdot & \cdot & \cdot & +, \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot, \\ + & + & + & + & \cdot & \cdot & \cdot & -. \end{array}$$

Again, if the points a and b give respectively

$$\begin{array}{cccccccc} - & + & + & + & \cdot & \cdot & \cdot & + \end{array}$$

and

$$\begin{array}{cccccccc} + & - & + & + & \cdot & \cdot & \cdot & +, \end{array}$$

then some point between a and b will make $L_1(x)$ vanish, and will give

$$\begin{array}{cccccccc} 0 & - & + & + & \cdot & \cdot & \cdot & +. \end{array}$$

This point must make $L_2(x)$ negative, as indicated, because we have shown that a point which makes any of the L's positive must make at least one L negative. Evidently, then, there exists a point which makes any arbitrarily chosen L vanish, any other one negative, and all the rest positive.

Between the two points which give respectively

$$\begin{array}{cccccccc} 0 & - & + & + & \cdot & \cdot & \cdot & + \end{array}$$

and

$$\begin{array}{cccccccc} 0 & + & - & + & \cdot & \cdot & \cdot & +, \end{array}$$

there is similarly some point which gives

$$\begin{array}{cccccccc} 0 & 0 & - & + & \cdot & \cdot & \cdot & +. \end{array}$$

By continuing this process, it is evident that we can establish the existence of a point p such that

$$L_i(p) = 0, \qquad i \neq s,\, t; \qquad L_s(p) < 0, \qquad \text{and} \qquad L_t(p) > 0,$$

L_s and L_t being any two of the L's chosen arbitrarily. Also, by carrying

the process one step further, it may be shown that there exists a point q such that

$$L_i(q) = 0, \qquad i \neq s; \qquad \text{and} \qquad L_s(q) \gtreqless 0.$$

If, now, there exists a set of constants $k_1, k_2, \cdots k_{m+1}$, not all zero, such that

$$\sum_{i=1}^{m} k_i L_i(x) + k_{m+1} \equiv 0,$$

it is evident that the identity

$$\sum_{i=1}^{m} k_i L_i'(x) \equiv 0$$

must also hold, and that k_{m+1} must be equal to $-\sum_{i=1}^{i=m} k_i \beta_i$. Since the system S is inconsistent, the rank of the matrix M must be less than m (by theorem 1); and hence it follows that there is at least one set of constants $k_1, k_2, \cdots k_m$, not all zero, such that

$$\sum_{i=1}^{m} k_i L_i'(x) \equiv 0.$$

If we first suppose that our system S is homogeneous, $L_i'(x) \equiv L_i(x)$, and we have

$$\sum_{i=1}^{m} k_i L_i(x) \equiv 0.$$

Substituting the point p in this identity, we have

$$k_s L_s(p) + k_t L_t(p) = 0;$$

and since $L_s(p) < 0$ and $L_t(p) > 0$, it follows that either k_s and k_t are both zero, or neither of them is zero and they have the same sign. But these are *any* two constants of the set $k_1, k_2, \cdots k_m$; and since not all of them are zero, none of them are zero and they all have the same sign. They may evidently all be made positive, and $k_1, k_2, \cdots k_m, 0$ is then such a set of constants as our theorem requires.

Suppose, on the other hand, that the system S is non-homogeneous. Then $L_i'(x) \equiv L_i(x) - \beta_i$, and we have

$$\sum_{i=1}^{m} k_i L_i(x) \equiv \sum_{i=1}^{m} k_i \beta_i.$$

Substituting the point q in this identity, we have

$$k_s L_s(q) = \sum_{i=1}^{m} k_i \beta_i.$$

If $\sum k_i \beta_i \neq 0$, then $k_s \neq 0$ and differs in sign from $\sum k_i \beta_i$. This means that none of the k's are zero, and that all of them have the sign contrary

to that of $\sum k_i\beta_i$. We may make them all positive, and with $k_{m+1} = - \sum k_i\beta_i$ we have a set of constants of the kind required by the theorem. For the case $\sum k_i\beta_i = 0$, none of the k's are zero, by the same argument that was used in the homogeneous case. Hence there must be a point q such that $L_i(q) = 0$, $i = 1, 2, \cdots m$. It follows that the transformation

$$x_j = x_j' + q_j, \qquad j = 1, 2, \cdots n,$$

sends this non-homogeneous system into the corresponding homogeneous system. And since such a transformation does not affect the existence or non-existence of solutions, the corresponding homogeneous system must be irreducibly inconsistent. It follows then, from our treatment of the homogeneous case, that the set of constants $k_1, k_2, \cdots k_m$ all have the same sign. Taking them all positive, and putting $k_{m+1} = 0$, we have such a set as the theorem requires.

We have then shown that for any system, homogeneous or non-homogeneous, which is irreducibly inconsistent, there exists at least one set of constants, $k_1, k_2, \cdots k_m$, not all zero, such that

$$\sum_{i=1}^{m} k_i L_i'(x) \equiv 0;$$

that in any such set none of the constants is zero, and all of them may be taken as positive; and that when we adjoin to any such set

$$k_{m+1} = - \sum_{i=1}^{m} k_i\beta_i$$

we then have such a set of k's as our theorem requires. It will follow that this set of constants is homogeneously unique when we show that the rank of the matrix M must be $m - 1$.

Suppose that the rank r of the matrix M were less than $m - 1$. Then for a properly chosen sub-set of $r + 1$ of the inequalities, say the first $r + 1$ of them, there would be a set of constants $f_1, f_2, \cdots f_{r+1}$, not all zero, such that

$$\sum_{i=1}^{r+1} f_i L_i'(x) \equiv 0.$$

These $r + 1$'s, together with $m - r - 1$ zeros, would make up a set of k's such that

$$\sum_{i=1}^{m} k_i L'(x) \equiv 0.$$

But we have shown that one of such a set of k's can not vanish unless they all vanish. Hence the rank of the matrix M can not be less than, and must therefore be equal to, $m - 1$. And it follows that the set of k's is homogeneously unique. This completes the proof of the theorem.

It is rather obvious that if solutions exist for a homogeneous system, they exist for any corresponding non-homogeneous system; and that the converse is not true.* But it follows from the proof of the last theorem that if a non-homogeneous system is irreducibly inconsistent, the same will be true of the corresponding homogeneous system.

Another by-product of the proof of the last theorem is the following fact: If in the matrix M of an irreducibly inconsistent system S we pick out any non-vanishing determinant of order $m - 1$, and throw out all the columns of the matrix except those involved in this determinant, we have left a matrix of $m - 1$ columns and m rows, in which the m determinants of order $m - 1$ alternate in sign, none of them vanishing.

THEOREM 3. *A necessary and sufficient condition that a given system S be inconsistent is that there should exist a set of $m + 1$ constants, k_1, k_2, $\cdots$ k_{m+1}, such that*

$$\sum_{i=1}^{m} k_i L_i(x) + k_{m+1} \equiv 0,$$

at least one of the k's being positive, and none of them being negative.

As to the sufficiency of the condition: suppose that a point a is a solution of the system, i.e., that $L_i(a) > 0$, $i = 1, 2, \cdots m$. Since at least one k is positive, and none are negative, it is obvious that the identity

$$\sum_{i=1}^{m} k_i L_i(x) + k_{m+1} \equiv 0$$

could not hold for this point. Hence there can be no solutions.

It remains to establish the necessity of the condition. If the system S is inconsistent, but not irreducibly inconsistent, we may drop out some inequality from the system which will leave an inconsistent sub-system of $m - 1$ inequalities. If this sub-system is not irreducibly inconsistent, we may drop one inequality from it, leaving an inconsistent sub-system of $m - 2$ inequalities. By continuing this process, we must finally arrive at an irreducibly inconsistent sub-system of ρ inequalities, where $1 \lesseqqgtr \rho \lesseqqgtr m$. We may think of this sub-system as consisting of the first ρ of the inequalities of our system S; and, by theorem 2, we have a set of constants $k_1, k_2, \cdots k_\rho, k_{m+1}$, such that

$$\sum_{i=1}^{\rho} k_i L_i(x) + k_{m+1} \equiv 0,$$

$k_1, k_2, \cdots k_\rho$ being positive, and k_{m+1} positive or zero. If now we put $k_{\rho+1} = k_{\rho+2} = \cdots = k_m = 0$, we have the set of constants required by our theorem.

In connection with the above proof it may be noted that an inconsistent

* Cf. Dines, loc. cit.

system S may have a number of different irreducibly inconsistent sub-systems. The rank of the matrix of any such sub-system of ρ inequalities is $\rho - 1$, and can not be greater than the rank r of the matrix M. Hence we must always have $\rho \leqq r + 1$. For a given inconsistent system, there may or may not be an irreducibly inconsistent sub-system containing as many as $r + 1$ inequalities.*

An inequality will be said to be *superfluous* in a system S, in which $m \geqq 2$, when it is satisfied by every point which satisfies all the other inequalities of the system.† In an inconsistent system, $m \geqq 2$, an inequality can be superfluous if and only if the sub-system obtained by omitting this inequality is inconsistent. We therefore have at once

THEOREM 4. *The necessary and sufficient condition that the inequality* $L_s(x) > 0$ *should be superfluous in an inconsistent system* S *is that there should exist a set of constants* $k_1, k_2, \cdots k_{m+1}$, *such that*

$$\sum_{i=1}^{m} k_i L_i(x) + k_{m+1} \equiv 0,$$

with $k_s = 0$, *at least one* k *positive, and none negative.*

THEOREM 5. *The necessary and sufficient condition that the inequality* $L_s(x) > 0$ *should be superfluous in a consistent system* S *is that there should exist a set of constants* $k_1, k_2, \cdots k_{m+1}$, *such that*

$$\sum_{i=1}^{m} k_i L_i(x) + k_{m+1} \equiv 0,$$

k_s *and no other* k *being negative, and at least one* k *being positive.*

The sufficiency of the condition is rather obvious. We have by hypothesis

$$L_s(x) \equiv \frac{k_1}{-k_s} L_1(x) + \cdots + \frac{k_{s-1}}{-k_s} L_{s-1}(x) + \frac{k_{s+1}}{-k_s} L_{s+1}(x) + \cdots + \frac{k_m}{-k_s} L_m(x) + \frac{k_{m+1}}{-k_s}.$$

* For instance, for the system

(1) $x_1 > 0$, (2) $x_2 > 0$, (3) $-2x_1 - x_2 - 5 > 0$, (4) $4x_1 + 2x_2 + 1 > 0$,

for which $r = 2$, if we drop (4), we have at once an irreducibly inconsistent system with $\rho = 3$; but if we first drop (1), we must then drop (2) before we arrive at an irreducibly inconsistent system with $\rho = 2$. Again, in the system

$$x_1 > 0, \quad x_2 - 1 > 0, \quad x_3 - 2 > 0, \quad -x_2 - x_3 + 1 > 0,$$

for which $r = 3$, we can drop only the first inequality, giving $\rho = 3$.

† For the case $m = 1$, we shall define an inequality to be superfluous in the system consisting of itself alone when and only when it is an identical inequality, i.e., when all the coefficients of the variables are zero and the constant term is positive. It is readily verified that the necessary and sufficient conditions of the next two theorems are in accord with this definition.

where at least one coefficient on the right is positive and none are negative. If in this identity we substitute a point a which satisfies all the inequalities of the system except possibly $L_s(x) > 0$, we see at once that $L_s(a)$ must also be positive.

To prove the necessity of the condition, consider a system S' obtained by replacing the inequality $L_s(x) > 0$ in S by the contradictory inequality $-L_s(x) > 0$. By hypothesis, every point which satisfies the inequalities of S other than $L_s(x) > 0$ must also satisfy this inequality, and hence can not satisfy the inequality $-L_s(x) > 0$. Hence S' is inconsistent, and there exists a set of constants $k_1, k_2, \cdots k_{m+1}$, such that

$$k_1L_1(x) + \cdots + k_{s-1}L_{s-1}(x) + k_s\{-L_s(x)\} + \cdots k_mL_m(x) + k_{m+1} \equiv 0,$$

at least one k being positive, and none negative. Moreover, we know that $k_s \neq 0$, and that at least one other k does not vanish, for otherwise the system S would be inconsistent. If then we replace k_s by $-k_s$, we have the set of constants required by the theorem.

A system S will be said to be *independent* if it contains no superfluous inequalities. In accordance with this definition, an irreducibly inconsistent system is an inconsistent system which is independent. A single inequality will always be independent except in the case of the identical inequality noted above.

Two systems may be said to be *equivalent* if every point which satisfies either of them satisfies the other one. Any two inconsistent systems are equivalent, and an inconsistent system can not be equivalent to a consistent system. A single inequality is obviously equivalent to another single inequality when and only when they are identically the same except possibly for a positive constant factor.

THEOREM 6. *If two systems S and $\sum$, each of which is independent and consistent, are equivalent, the number of inequalities in the two systems is the same, and each inequality of one system is equivalent to one and only one inequality of the other system; i.e., the inequalities of the two systems are identical except for possible positive constant factors.*

Let $L_s(x) > 0$ be any inequality of the system S. Since it is not superfluous in S, and S is consistent, there must exist a point a such that $L_i(a) > 0$, $i \neq s$, and $L_s(a) \overline{\underline{<}} 0$; and also a point b such that $L_i(b) > 0$, $i = 1, 2, \cdots m$. Hence there must be a point c coincident with a or between a and b, such that $L_i(c) > 0$, $i \neq s$, and $L_s(c) = 0$. Since there is one such point, there must be an infinite number of them; for every point satisfying the equation $L_s(x) = 0$ and lying in a sufficiently small region about c will satisfy the same conditions. Let G represent the set of all points satisfying these conditions, $L_i(c) > 0$, $i \neq s$, and $L_s(c) = 0$. Let H represent the set of all points satisfying the system S. The only

limit points of H which do not belong to H are points which satisfy the equations $L_i(x) = 0$ for one or more values of i, and the inequalities $L_i(x) > 0$ for the remaining values of i. The points of the set G are such limit points of H. But since, by hypothesis, H is also the set of all points satisfying the system $\sum$, each point of the set G must satisfy at least one equation $\lambda_i(x) = 0$ corresponding to an inequality $\lambda_i(x) > 0$ of the set $\sum$. And since there are only a finite number of inequalities in the set $\sum$, at least one equation, say $\lambda_s(x) = 0$, must be satisfied by an infinite number of points of G. Hence the equation $\lambda_s(x) = 0$ must be equivalent to the equation $L_s(x) = 0$; and the inequality $\lambda_s(x) > 0$ must be equivalent to the inequality $L_s(x) > 0$. Moreover, an inequality of S can not be equivalent to more than one inequality of $\sum$, for in that case these inequalities in $\sum$ would all be equivalent to each other, and all but one of them would be superfluous in $\sum$.

If one drops a superfluous inequality from a consistent system S, the remaining system of $m - 1$ inequalities is evidently equivalent to the original system. If this system of $m - 1$ inequalities is not independent, a superfluous inequality may be dropped from it. By continuing this process, we must finally arrive at an independent sub-system equivalent to the original system.* The order in which the superfluous inequalities are dropped in this process is immaterial; for, by the last theorem, any two independent sub-systems obtained in this way can differ only by positive constant factors in the inequalities. This is in distinct contrast to the facts for an inconsistent system.

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* The only exception is the trivial case in which all the inequalities of the system S are the identical inequalities noted above.